Dynamics 4

Learning Module 5

Longitudinal waves



THE UNIVERSITY *of* EDINBURGH
School of Engineering

DYNAMICS 4

LM5 GUIDE 26 27

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Learning Module 5 | Longitudinal waves

Estimated time: 14 hours Complete module before: 11 November

1. Start by reading the Course Notes: For this module you should read **Sections 3.1 and 3.2.** Take hand-written notes as you read.

2. Attend the lectures and tutorial sessions and, if you need, watch the lecture recordings later. This will help consolidate what you learned. Keep taking notes.

3. Attempt to solve **Questions 5.1 to 5.6** in **Exercise Sheet 5**. Compare your solutions to the numerical solutions provided at the end of each question. Write down any questions you might have.

4. Compare your solutions to the worked-out example solutions to **Questions 5.1 to 5.6**. Discuss any questions you might have in the tutorial sessions. Arrange to meet with your lecturer if you need further clarifications.

5. Be active in the Learn discussion board thread for this Learning Module, which is labeled **longitudinal_waves**. You can post your own questions and help clarify or answer the questions of others.

6. Have a look at the additional resources provided in the **Resources folder** for Learning Module 5, on Learn. You should also read **Section 3.4** of the Course Notes with examples of real life applications.



In this module

Dynamics of continuous systems
Longitudinal waves in solids
Travelling and standing wave solutions
Boundary conditions

Learning outcomes

Derive equations of motion for dynamic systems from energy-based approaches (e.g. Lagrange's method)

Know the common wave equations for basic structural elements and be able to use these to better understand the response of simple structural elements

This is a study and learning guide to Learning Module 5. You can find the corresponding lecture recordings and slides on Learn. This module has a dedicated thread in the discussion board on Learn.

Please note that the course time and the recommended *before dates*, are estimated and may differ from person to person. You should not see these as targets but instead as a way to help you plan your semester and the preparation for exams. The estimated time includes going through all the course materials, interaction in the discussion boards and self-study time.

Wave equations

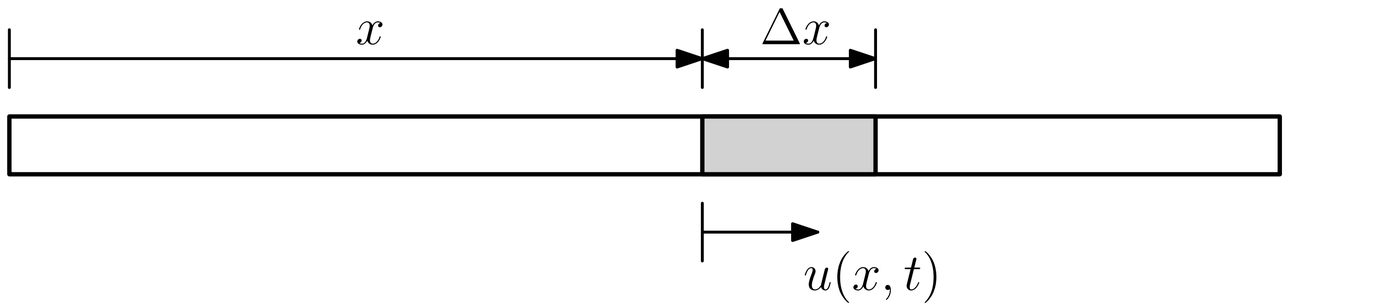
- Continuous bodies
- Displacement calculated at every point (displacement field)
  - Distributed mass
  - Distributed elasticity and stiffness
  - Infinite number of degrees-of-freedom
  - Described by partial differential equations
  - Boundary conditions are of major importance

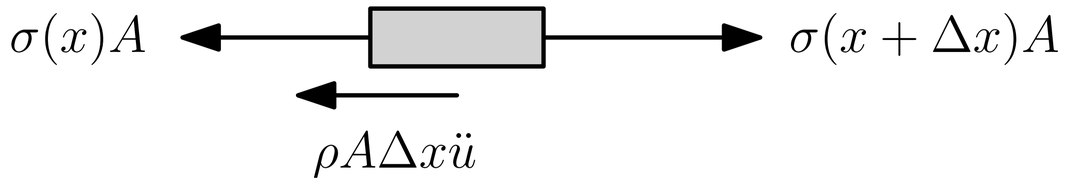
Wave equations

- Only uniform cases can be treated easily
- Bars, beams and membranes
  - Travelling and standing waves
  - Axial and longitudinal waves (e.g. sound waves)
  - Flexural (bending) waves in thin beams
  - Dispersion
  - Natural frequencies and mode shapes

Longitudinal (axial) waves

- Equilibrium considerations





Wave equations

- Longitudinal (axial) waves:

$$\frac{\partial^2 u}{\partial t^2} = \frac{E}{\rho}\frac{\partial^2 u}{\partial x^2}$$

- Transverse waves:

$$\frac{\partial^2 w}{\partial t^2} = \frac{T}{\mu}\frac{\partial^2 w}{\partial x^2}$$

- Torsional waves (circular bar):

$$\frac{\partial^2 \theta}{\partial t^2} = \frac{G}{\rho}\frac{\partial^2 \theta}{\partial x^2}$$

Wave equations

Some displacement field

Generic form

$$\frac{\partial^2 \Phi}{\partial t^2} = c^2 \frac{\partial^2 \Phi}{\partial x^2}$$

The wave speed in the medium (1D speed of sound)

Wave equations

Generic form

$$\frac{\partial^2 \Phi}{\partial t^2} = c^2 \frac{\partial^2 \Phi}{\partial x^2}$$

Travelling wave solutions

$$u(x,t) = A \sin\left[\frac{2\pi}{\lambda}(x - ct)\right]$$

Wave speed

Wave amplitude

Wavelength

Wave equations

- Travelling wave solutions

$$u(x,t) = A\sin\left[\frac{2\pi}{\lambda}(x-ct)\right]$$

- Inserting the wave equation yields

$$-\frac{4\pi^2}{\lambda^2}c^2 A\sin\left[\frac{2\pi}{\lambda}(x-ct)\right] = \frac{E}{\rho}\left(-\frac{4\pi^2}{\lambda^2}\right)A\sin\left[\frac{2\pi}{\lambda}(x-ct)\right]$$

$$\Rightarrow c^2 = \frac{E}{\rho} \quad \left(\omega = \frac{2\pi c}{\lambda} \quad \text{and} \quad f = \frac{c}{\lambda}\right)$$

Non-dispersive waves

- Wave speed not dependent on wavelength

$$\Rightarrow c^2 = \frac{E}{\rho} \quad \left(\omega = \frac{2\pi c}{\lambda} \quad \text{and} \quad f = \frac{c}{\lambda}\right)$$

- Travel at constant speed and without change in shape

Standing wave solutions

- Longitudinal (axial) waves:

$$\frac{\partial^2 u}{\partial t^2} = \frac{E}{\rho}\frac{\partial^2 u}{\partial x^2}$$

Assume product of space/time-dependent functions:

$$u(x,t) = X(x)T(t)$$

yields

$$X\frac{\partial^2 T}{\partial t^2} = c^2 T\frac{\partial^2 X}{\partial x^2} \Rightarrow \frac{1}{T}\frac{\partial^2 T}{\partial t^2} = c^2\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = -\omega^2$$

Standing wave solutions

- Two resulting equations:

$$\frac{\partial^2 T}{\partial t^2} + \omega^2 T = 0 \quad \text{and} \quad \frac{\partial^2 X}{\partial x^2} + \frac{\omega^2}{c^2} X = 0$$

with solutions:

$$T(t) = C_1 \cos(\omega t) + C_2 \sin(\omega t)$$

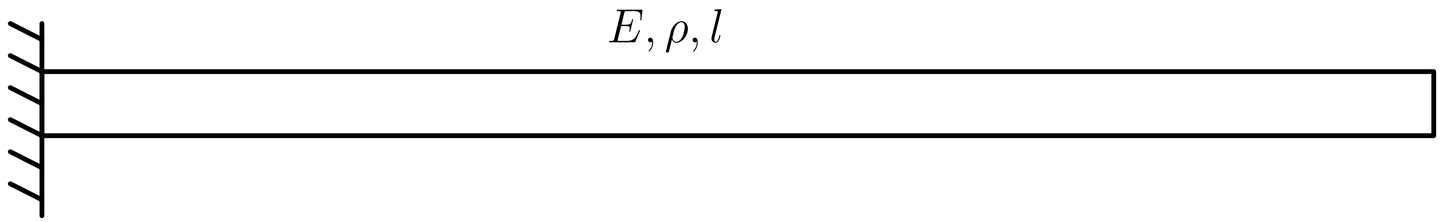
$$X(x) = C_3 \cos\left(\frac{\omega}{c} x\right) + C_4 \sin\left(\frac{\omega}{c} x\right)$$

or:

$$T(t) = A \cos(\omega t) \quad \text{and} \quad X(x) = \cos\left(\frac{\omega}{c} x\right)$$

Example (CN)

Determine the possible longitudinal (axial) vibrations of the uniform bar shown, with one clamped and one free end.



Example (CN)

Calculate first two frequencies of axial vibration and determine mode shapes of the bar shown. Compare with single degree-of-freedom lumped parameter model. Model parameters: length l = 3.5 m, mass M = 50 kg, cross section 50 x 50 [mm^2], elastic modulus E = 207 GPa and density ρ = 7830 kg/m^3.

